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A simple criterion for similar and opposite orderings

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ABSTRACT. In this paper, a simple criterion is given for similar and opposite orderings of real n-tuples.

ABSTRAK. Dalam kertas ini, suatu kriteria mudah diberi bagi pentertiban berserupaan dan bertentangan rangkap-rangkap-n nyata.

(similar/opposite orderings)

INTRODUCTION

In an inequality of Tchebychef [1, Theorem 43, p. 43] and also in an inequality of Hardy, Littlewood and Pólya [1, Theorem 368, p. 261], the concept of similar and opposite orderings is used. In [1, Theorem 369, p. 262], Hardy, Littlewood and Pólya gave a sufficient condition for a pair of n-vectors of real numbers to be similarly ordered by means of rearrangements of n-tuples. In practice, it is obvious that neither the condition of Hardy, Littlewood and Pólya nor the very definition itself (see (3) below) can be easily applied to checking for the similar or opposite orderliness of a pair of real n-vectors. In this paper, a simple and easily applicable necessary and sufficient condition is given for two real n-tuples to be similarly or oppositely ordered.

PRELIMINARIES

In the sequel, we let R^n denote the set of all n-tuples of real numbers. For any n-tuple $x = (x_1, x_2, ..., x_n) \in R^n$, we denote by $x = (x^*_1, x^*_2, ..., x^*_n)$ and $x' = (x'_1, x'_2, ..., x'_n)$ the n-tuples whose components are respectively those of x arranged in non-increasing and non-decreasing order of magnitude. Moreover, if $\mathbf{a}, \mathbf{b} \in R^n$, then we write $\mathbf{a} \sim \mathbf{b}$ to mean that the components of $\mathbf{a}$ form a rearrangement of those of $\mathbf{b}$. Thus, $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a}^* = \mathbf{b}^*$. Furthermore, if $\mathbf{a} = (a_1, a_2, ..., a_n) \sim (b_1, b_2, ..., b_n) = \mathbf{b}$ and if $\mathbf{c} = (c, c, ..., c) \in R^n$ is any constant vector, then it is clear that

(1) $$\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

i.e., $(a_1 + c, a_2 + c, ..., a_n + c) \sim (b_1 + c, b_2 + c, ..., b_n + c)$.

If $\mathbf{x}_i \in R_n$ are n_i-tuples, $i = 1, 2, ..., m$, let $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m)$, be the $(n_1 + n_2 + ... + n_m)$-tuple whose components are those of $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m$. Then, if $\mathbf{a}_i \sim \mathbf{b}_i$, $i = 1,2, ..., m$, it is obvious that

(2) $$(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_m) \sim (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m) .$$

n-tuples $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in R^n are said to be **similarly ordered** if

(3) $$(a_i - a_j)(b_i - b_j) \geq 0$$

for all $i, j = 1,2, ..., n$, and **oppositely ordered** if the inequality (3) is always reversed.

A SIMPLE CRITERION

The following theorem gives a simple characterization for a pair of n-tuples in R^n to be similarly or oppositely ordered.

THEOREM Two n-tuples $\mathbf{a}$ and $\mathbf{b}$ in R^n are similarly (respectively oppositely) ordered if and only if $\mathbf{a} + \mathbf{b} \sim \mathbf{a}^* + \mathbf{b}^*$ (*respectively* $\mathbf{a} + \mathbf{b} \sim \mathbf{a}^* + \mathbf{b}'$).

PROOF To prove the sufficiency of the condition, suppose that $\mathbf{a} + \mathbf{b} \sim \mathbf{a}^* + \mathbf{b}^*$. Then there exists a permutation π of the integers 1,2, ..., n such that

(4) $$a_{\pi(i)} + b_{\pi(i)} = a^*_i + b^*_i$$

for $i = 1,2, ..., n$. Now $a_{\pi(1)} \leq a^*_1$ and $b_{\pi(1)} \leq b^*_1$ and so $a_{\pi(1)} = a^*_1$ and $b_{\pi(1)} = b^*_1$ by virtue of (4) with $i = 1$. But then $a_{\pi(2)} \leq a^*_2$ and $b_{\pi(2)} \leq b^*_2$ so $a_{\pi(2)} = a^*_2$ and $b_{\pi(2)} = b^*_2$ in view of (4) again with $i = 2$. Clearly, the same argument can be repeated in turn for $i = 3,4, ..., n$ to conclude that $a_{\pi(i)} = a^*_i$ and $b_{\pi(i)} = b^*_i$, $i = 1,2, ..., n$. Thus $\mathbf{a}$ and $\mathbf{b}$ are similarly ordered since

$$(a_i - a_j)(b_i - b_j) = (a^*_{\sigma(i)} - a^*_{\sigma(j)})(b^*_{\sigma(i)} - b^*_{\sigma(j)}) \geq 0$$

where $\sigma = \pi^{-1}$ denotes the inverse of π and $i, j = 1,2, ...,n$.

To prove the necessity of the condition, suppose that **a** and **b** are similarly ordered. Assume that the sequence { $a^*_1, a^*_2, ..., a^*_n$} is "constant" except for the integers $k_1, k_2, ..., k_m$ such that $1 \leq k_1 < k_2 < ... < k_m \leq n$, i.e., $a^*_{k_1} > a^*_{k_2} > ... > a^*_{k_m}$ and

$$(5) \quad \begin{cases} a^*_1 = a^*_2 = ... = a^*_{k_1-1} > a^*_{k_1}, & \text{if } k_1 < 1 \\ a^*_{k_m-1} > a^*_{k_m} = a^*_{k_m+1} = ... a^*_n, & \text{if } k_m < n \\ a^*_{k_j} = a^*_{k_j+1} = ... a^*_{k_{j+1}-1} & \text{for } j = 1, 2, ..., m\text{-}1. \end{cases}$$

Now it is easy to see that there exists a permutation π of the integers 1,2, ..., n such that

$$a_{\pi(i)} = a^*_i \qquad \text{for } i = 1,2, ..., n.$$

Since **a** and **b** are similarly ordered, it follows from (4) and (5) that

$$(6) \qquad b_{\pi(p)} \geq b_{\pi(q)}$$

for $(p,q) \in [\{1, ..., k_1 -1\} \times \{k_1, ..., k_2 -1\}] \bigcup_{j=1}^{m-3} [\{k_j, ..., k_{j+1} -1\} \times \{k_{j+1}, ..., k_{j+2}-1\}] \cup [\{k_{m-1}, ..., k_m-1\} \times \{k_m, ..., n\}]$.

In view of (6), it is clear that if

$\mathbf{b}_1 = (b_{\pi(1)}, ..., b_{\pi(k_1-1)})$,

$\mathbf{b}_j = (b_{\pi(k_{j-1})}, ..., b_{\pi(k_j-1)})$, $j = 2, ..., m$ and

$\mathbf{b}_{m+1} = (b_{\pi(k_m)}, ..., b_{\pi(n)})$,

then

$\mathbf{b}^*_1 = (b^*_1, ..., b^*_{k_1-1})$,

$\mathbf{b}^*_j = (b^*_{k_{j-1}}, ..., b^*_{k_j-1})$, $j = 2, ..., m$, and

$\mathbf{b}^*_{m+1} = (b^*_{k_m}, ..., b^*_n)$.

Now let

$\mathbf{a}^*_1 = (a^*_1, ..., a^*_{k1-1})$,

$\mathbf{a}^*_j = (a^*_{k_{j-1}}, ..., a^*_{k_j-1})$, $j = 2, ..., m$, and

$\mathbf{a}^*_{m+1} = (a^*_{k_m}, ..., a^*_n)$.

Then

$$\mathbf{a} + \mathbf{b} \sim (a_{\pi(1)} + b_{\pi(1)}, ..., a_{\pi(n)} + b_{\pi(n)})$$
$$= (a^*_1 + b_{\pi(1)}, ..., a^*_n + b_{\pi(n)})$$
$$= (\mathbf{a}^*_1 + \mathbf{b}_1, ..., \mathbf{a}^*_{m+1} + \mathbf{b}_{m+1}) \sim$$
$$(\mathbf{a}^*_1 + \mathbf{b}^*_1, ..., \mathbf{a}^*_{m+1} + \mathbf{b}^*_{m+1})$$
$$= \mathbf{a}^* + \mathbf{b}^*$$

by virtue of (1), (2) and (5).

The proof for the case that **a** and **b** are oppositely ordered is similar.

REFERENCE

1 G.H. Hardy, J.E. Littlewood and G. Pólya (1959). *Inequalities*. Cambridge University Press, London and New York.