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SOLUTION OF ORDINARY DIFFERENTIAL EQUATION v^{vi} (u)=f(u,v,v',v'',v''') USING EIGHTH AND NINTH ORDER RUNGE-KUTTA TYPE METHOD

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Abstract: The present paper presents the numerical conclusion to solve sixth order initial value ordinary differential equation (ODE). The concept of order conditions for three stage eighth order (RKSD8) & four stage ninth order Runge-Kutta methods (RKSD9) has been derived for finding global truncation error of differential equation v^{vi} = f(u,v,v', v'',v '''). The global and local truncated errors norms, zero stability of extended Runge-Kutta method (RK) is well defined and demonstrated with the help of an example.

Keywords: *Ordinary differential equations (ODE), runge-kutta type methods, local and global truncation error, zero stability*

1. Introduction

The present article works upon use of Runge-Kutta type method for solving the sixth order ordinary differential equations (ODE). According to literature review, differential equations are extensively utilized in science and engineering for mathematical modeling. Many mathematical physics issues can be stated with the aid of differential equations [13]. Hatun M et al [6] initiated ODE solver simulator for RK method. Khalid et al. [9], Pandey et al. [11] applied finite difference approach and neural network concept for examining ODE problems. The diagonally implicit fifth and sixth order RK method been initiated by Abbas F et al. [1] in 2017. Various properties such as stability of second order differential equations have also been worked upon by Mohamed et al [10]. Sohaib, et al. [12] applied Legendre wavelet collocation technique for 6th order problems. In 2021,Huang, B. et al [7] had also evaluated error analysis with implicit–explicit RK-Rosenbrock method. Islam et al [8] had done effective comparison on numerical solutions of initial value problems for ODE with Euler and RK methods. In 2021, Abdi et al [2] had analyzed global error estimation for explicit problems. The present research proposed the comprehensive approach for solving sixth order $v^{vi} = f(u, v, v'\ v'', v''')$to achieve more accuracy and stability using RKSD.

2. Runge-Kutta Type Sixth Order Ode

The initial value problem of sixth order ODE examined in the present paper are:

$$v^{vi}(u) = f(u, v, v', v'', v''') \qquad [1]$$

possessing initial conditions as

$$v(u_0) = \alpha_0, v'(u_0) = \alpha_0', v''(u_0) = \alpha_0'', \\ v'''(u_0) = \alpha_0''', v^{iv}(u_0) = \alpha_0^{iv}, v^{v}(u_0) = \alpha_0^{v} \qquad [2]$$

in which derivatives of fourth and fifth order do not appear explicitly. In this section, we present the general form of initial value problem Equation (1) as follows:

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$$v_{n+1} = v_n + hv_n' + \frac{h^2}{2}v_n'' + \frac{h^3}{3!}v_n''' + \frac{h^4}{4!}v_n^{iv} + \frac{h^5}{5!}v_n^{v} + h^6\sum_{i=1}^{s} b_i k_i \quad [3]$$

$$v_{n+1}' = v_n' + hv_n'' + \frac{h^2}{2}v_n''' + \frac{h^3}{3!}v_n^{iv} + \frac{h^4}{4!}v_n^{v} + h^5\sum_{i=1}^{s} b_i' k_i \quad [4]$$

$$v_{n+1}'' = v_n'' + hv_n''' + \frac{h^2}{2}v_n^{iv} + \frac{h^3}{3!}v_n^{v} + h^4\sum_{i=1}^{s} b_i'' k_i \quad [5]$$

$$v_{n+1}''' = v_n''' + hv_n^{iv} + \frac{h^2}{2}v_n^{v} + h^3\sum_{i=1}^{s} b_i''' k_i \quad [6]$$

$$v_{n+1}^{iv} = v_n^{iv} + hv_n^{iv} + h^2\sum_{i=1}^{s} b_i^{iv} k_i \quad [7]$$

$$v_{n+1}^{v} = v_n^{v} + h\sum_{i=1}^{s} b_i^{v} k_i \quad [8]$$

where

$$k_1 = f\left(u_n, v_n, v_n', v_n'', v_n'''\right)$$

$$k_i = f\Bigg(u_n + c_i h, v_n + hc_i v_n' + \frac{h^2c_i^2}{2}v_n'' + \frac{h^3c_i^3}{3!}v_n''' + \frac{h^4c_i^4}{4!}v_n^{iv} + \frac{h^5c_i^5}{5!}v_n^{v} + h^6\sum_{j=1}^{s} a_{ij}k_j, v_n' + hc_i v_n'' + \frac{h^2c_i^2}{2}v_n''' + \frac{h^3c_i^3}{3!}v_n^{iv} + \frac{h^4c_i^4}{4!}v_n^{v} + h^5\sum_{j=1}^{s} \overline{a_{ij}}k_j, v_n'' + hc_i v_n''' + \frac{h^2c_i^2}{2}v_n^{iv} + \frac{h^3c_i^3}{3!}v_n^{v} + h^4\sum_{j=1}^{s} \overline{\overline{a_{ij}}}k_j, v_n''' + hc_i v_n^{iv} + \frac{h^2c_i^2}{2}v_n^{v} + h^3\sum_{j=1}^{s} \overline{\overline{\overline{a_{ij}}}}k_j\Bigg) \quad for \quad i = 1,2,3,\ldots,s.$$

The pivotal condition for RK method for sixth order differential equation is that the parameters $a_{ij}, b_i, b_i', b_i'', b_i''', b_i^{iv}, b_i^{v}$ and c_i should be real for all $i, j = 1, 2, 3, \ldots, s$. The method will be treated explicit one if $a_{ij} = 0$ at $i > j$ however if $a_{ij} \neq 0$ at $i < j$ the method change its behavior to implicit one.

The Taylor's series concept been applied for finding relations to various parameters associated with RK method in equations (3)-(8). The main objective of research is constructing an embedded pair of the explicit RKSD methods for evaluating low value local truncated error as done by [3]–[5], [14-18] which will be useful in step-size algorithm. Method will calculate the value to the parameter $v_{n+1}, v_{n+1}', v_{n+1}'', v_{n+1}''', v_{n+1}^{iv}, v_{n+1}^{v}$ for obtaining approximate value to $v(u_{n+1}), v'(u_{n+1}), v''(u_{n+1}), v'''(u_{n+1})$, $v^{iv}(u_{n+1}), v^{v}(u_{n+1})$ where v_{n+1} is the calculated solution and $v(u_{n+1})$is the original solution. Equations (3)-(8) can be represented as:

$$v_{n+1} = v_n + h\psi \quad [10]$$

$$v_{n+1}' = v_n' + h\psi' \quad [11]$$

$$v_{n+1}'' = v_n'' + h\psi'' \quad [12]$$

$$v_{n+1}''' = v_n''' + h\psi''' \quad [13]$$

$$v_{n+1}^{iv} = v_n^{iv} + h\psi^{iv} \quad [14]$$

$$v_{n+1}^{v} = v_n^{v} + h\psi^{v} \quad [15]$$

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The differentials for the scalar equation are:

$$F_1^{(6)} = v^{(vi)} = f(u, v, v_n', v_n'', v_n''', v_n^{iv}), \tag{16}$$

$$F_1^{(7)} = g(u, v, v_n', v_n'', v_n''', v_n^{iv}) = f_u + f_v v' + f_{v'} v_{uu} + f_{v''} v_{uuu} + f_{v'''} v_{uuuu} + f_{v^{iv}} v_{uuuuu} \tag{17}$$

$$F_1^{(8)} = g_u + g_v\ v' + g_{v'} v_{uu} + g_{v''} v_{uuu} + g_{v'''} v_{uuuu} + g_{v^{iv}} v_{uuuuu} \tag{18}$$

Δ is assumed to be a Taylor series increment function for which the local truncation error of $v(u), v'(u), v''(u), v'''(u),\ v^{iv}(u),\ v^{v}(u)$ is calculated from [9-15].

$$\tau_{n+1}^{p} = h[\psi^{p} - \Delta^{\mathrm{p}}], \qquad where\ p = (0), (i), (ii), \dots\dots, (v) \tag{19}$$

Using equations [16-18], the Taylor series increments functions of $v(u), v'(u), v''(u), v'''(u), v^{iv}(u), v^{v}(u)$ can be expressed as:

$$\Delta = v_n{}' + \frac{1}{2} h v_n{}'' + \frac{1}{3!} h^2 v_n{}''' + \frac{1}{4!} h^3 v_n{}^{iv} + \frac{1}{5!} h^4 v_n{}^{v} + \frac{1}{6!} h^5 F_1^{(6)} + O(h^6) \tag{20}$$

Similarly, solving the same for other Δ function as $\Delta', \Delta'', \Delta''', \Delta^{\mathrm{iv}}, \Delta^{\mathrm{v}}$. Using Equations (11)-(15) and Equations (16), the local truncation errors, Equation (19) will be represented as:

$$\tau_{n+1} = h^6 \left[\sum b_i k_i - \left(\frac{1}{6!} F_1^{(6)} + \frac{1}{7!} h F_1^{(7)} + \frac{1}{8!} h\ ^2F_1^{(8)} + \frac{1}{9!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{21}$$

$$\tau_{n+1}' = h^5 \left[\sum b_i' k_i - \left(\frac{1}{5!} F_1^{(6)} + \frac{1}{6!} h F_1^{(7)} + \frac{1}{7!} h\ ^2F_1^{(8)} + \frac{1}{8!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{22}$$

$$\tau_{n+1}'' = h^4 \left[\sum b_i'' k_i - \left(\frac{1}{4!} F_1^{(6)} + \frac{1}{5!} h F_1^{(7)} + \frac{1}{6!} h\ ^2F_1^{(8)} + \frac{1}{7!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{23}$$

$$\tau_{n+1}''' = h^3 \left[\sum b_i''' k_i - \left(\frac{1}{3!} F_1^{(6)} + \frac{1}{4!} h F_1^{(7)} + \frac{1}{5!} h\ ^2F_1^{(8)} + \frac{1}{6!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{24}$$

$$\tau_{n+1}^{iv} = h^2 \left[\sum b_i^{iv} k_i - \left(\frac{1}{2!} F_1^{(6)} + \frac{1}{3!} h F_1^{(7)} + \frac{1}{4!} h\ ^2F_1^{(8)} + \frac{1}{5!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{25}$$

$$\tau_{n+1}^{v} = h \quad \left[\sum b_i^{v} k_i - \left(F_1^{(6)} + \frac{1}{2!} h F_1^{(7)} + \frac{1}{3!} h\ ^2F_1^{(8)} + \frac{1}{4!} h\ ^3F_1^{(9)} \dots\dots\dots\right)\right] \tag{26}$$

The most important precondition for obtaining the convergence of numerical problem is evaluating zero-stability of the system, as explained by Dormand et al. [3]. The method used in current paper be written in an array form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ h v_{n+1}' \\ h^2 v_{n+1}'' \\ h^3 v_{n+1}''' \\ h^4 v_{n+1}^{iv} \\ h^5 v_{n+1}^{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_n \\ h v_n' \\ h^2 v_n'' \\ h^3 v_n''' \\ h^4 v_n^{iv} \\ h^5 v_n^{v} \end{bmatrix}$$

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The characteristic equation will be expressed as:

$$\rho(\varepsilon) = |I\varepsilon - A|$$

Hence, $\rho(\varepsilon) = (\varepsilon - 1)^6$ we get the roots to be $\varepsilon = 1,1,1,1,1,1$ which proves the desired zero stability of the proposed method as no root is found to be greater than 1 and also the multiplicity of the roots is at most 6.

2.2 RKSD8 and RKSD9 Method Order Conditions:

The order conditions derived for the three and four stage RKSD8 and RKSD9 are obtained as:

The order terms for v are:

$$\text{Sixth order:} \sum b_i = \frac{1}{6!} = \frac{1}{720}, \quad \text{Seventh order:} \sum b_i c_i = \frac{1}{7!} = \frac{1}{5040},$$

$$\text{Eighth order: } \sum b_i c_i^2 = \frac{1}{20160}, \text{ Ninth order: } \sum b_i c_i^3 = \frac{1}{60480} \qquad [27\text{-}30]$$

The order terms for v' are:

$$\text{Fifth order:} \sum b_i' = \frac{1}{5!} = \frac{1}{120}, \quad \text{Sixth order:} \sum b_i' c_i = \frac{1}{6!} = \frac{1}{720},$$

$$\text{Seventh order:} \sum b_i' c_i^2 = \frac{1}{2520}, \quad \text{Eighth order:} \sum b_i' c_i^3 = \frac{1}{6720}, \sum b_i' c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{40320}$$

$$\text{Ninth Order:} \sum b_i' c_i^4 = \frac{1}{15120} \qquad [31\text{-}35]$$

The order terms for v'' are:

$$\text{Fourth order:} \sum b_i'' = \frac{1}{4!} = \frac{1}{24}, \text{ Fifth order:} \sum b_i'' c_i = \frac{1}{5!} = \frac{1}{120},$$

$$\text{Sixth order:} \sum b_i'' c_i^2 = \frac{1}{360}, \quad \text{Seventh order:} \sum b_i'' c_i^3 = \frac{1}{840}, \sum b_i'' c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5040}$$

$$\text{Eighth order: } \sum b_i'' c_i^4 = \frac{1}{1680}, \sum b_i'' c_j \overline{\overline{a_{ij}}} = \frac{1}{40320}, \sum b_i'' c_i \overline{\overline{a_{ij}}} = \frac{1}{10080}, \sum b_i'' \overline{\overline{a_{ij}}} = \frac{1}{40320}$$

$$\text{Ninth Order: } \sum b_i'' c_i^5 = \frac{1}{3024} \qquad [36\text{-}45]$$

The order terms for v''' are:

$$\text{Third Order:} \sum b_i''' = \frac{1}{3!} = \frac{1}{6}, \quad \text{Fourth order:} \sum b_i''' c_i = \frac{1}{4!} = \frac{1}{24}$$

$$\text{Fifth order:} \sum b_i''' c_i^2 = \frac{1}{60}, \quad \text{Sixth order:} \sum b_i''' c_i^3 = \frac{1}{120}, \sum b_i''' \overline{\overline{\overline{a_{ij}}}} = \frac{1}{720}$$

$$\text{Seventh order:} \sum b_i''' c_i^4 = \frac{1}{210}, \sum b_i''' c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5040} \sum b_i''' c_i \overline{\overline{a_{ij}}} = \frac{1}{1260}, \sum b_i''' \overline{\overline{a_{ij}}} = \frac{1}{5040}$$

$$\text{Eighth Order: } \sum b_i''' c_i^5 = \frac{1}{336}, \sum b_i''' c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{20160}, \sum b_i''' c_i^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{2016}, \sum b_i''' c_i c_j \overline{\overline{a_{ij}}} = \frac{1}{8064},$$

$$\sum b_i''' c_i \overline{\overline{a_{ij}}} = \frac{1}{8064}, \sum b_i''' c_j \overline{\overline{a_{ij}}} = \frac{1}{40320}, \sum b_i''' \overline{a_{ij}} = \frac{1}{40320}$$

$$\text{Ninth Order: } \sum b_i''' c_i^6 = \frac{1}{504} \qquad [46\text{-}62]$$

The order terms for v^{iv} are:

$$\text{Second Order:} \sum b_i^{iv} = \frac{1}{2}, \quad \text{Third order:} \sum b_i^{iv} c_i = \frac{1}{3!} = \frac{1}{6},$$

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Fourth order: $\sum b_i^{iv} c_i^2 = \frac{1}{12}$, Fifth order: $\sum b_i^{iv} c_i^3 = \frac{1}{20}, \sum b_i^{iv} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{120}$

Sixth order: $\sum b_i^{iv} c_i^4 = \frac{1}{30}, \sum b_i^{iv} c_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{180}, \sum b_i^{iv} c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{720}, \sum b_i^{iv} \overline{\overline{a_{ij}}} = \frac{1}{720}$

Seventh order: $\sum b_i^{iv} c_i^5 = \frac{1}{42}, \sum b_i^{iv} c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{2520}, \sum b_i^{iv} c_i^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{252}, \sum b_i^{iv} c_i\, c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{1008}$

$$\sum b_i^{iv} c_j \overline{\overline{a_{ij}}} = \frac{1}{5040}, \sum b_i^{iv} c_i \overline{\overline{a_{ij}}} = \frac{1}{1008}, \sum b_i^{iv} \overline{a_{ij}} = \frac{1}{5040}$$

Eighth order: $\sum b_i^{iv} c_i^6 = \frac{1}{56}, \sum b_i^{iv} c_j^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{6720}, \sum b_i^{iv} c_i^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{336}, \sum b_i^{iv} c_i^2 c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{1344},$

$$\sum b_i^{iv} c_i\, c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{6720}, \sum b_i^{iv} c_j \overline{\overline{a_{ij}}} = \frac{1}{40320}, \sum b_i^{iv} c_i \overline{a_{ij}} = \frac{1}{6720}, \sum b_i^{iv} a_{ij} = \frac{1}{40320}, \sum b_i^{iv} c_j^2 \overline{\overline{a_{ij}}} = \frac{1}{20160},$$

$\sum b_i^{iv} c_i^2 \overline{\overline{a_{ij}}} = \frac{1}{1344}, \sum b_i^{iv} c_i\, c_j \overline{\overline{a_{ij}}} = \frac{1}{6720}$, Ninth Order: $\sum b_i^{iv} c_i^7 = \frac{1}{72}$ [63-90]

The order terms for v^v are:

First Order: $\sum b_i^v = 1$, Second order: $\sum b_i^v c_i = \frac{1}{2}$,

Third order: $\sum b_i^v c_i^2 = \frac{1}{3}$, Fourth order: $\sum b_i^v c_i^3 = \frac{1}{4}, \sum b_i^v \overline{\overline{\overline{a_{ij}}}} = \frac{1}{24}$

Fifth order: $\sum b_i^v c_i^4 = \frac{1}{5}, \sum b_i^v \overline{\overline{a_{ij}}} = \frac{1}{120}, \sum b_i^v c_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{30}, \sum b_i^v c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{120}$

Sixth order: $\sum b_i^v c_i^5 = \frac{1}{6}, \sum b_i^v c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{360}, \sum b_i^v c_i^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{36}, \sum b_i^v c_i\, c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{144},$

$$\sum b_i^v c_j \overline{\overline{a_{ij}}} = \frac{1}{720}, \sum b_i^v c_i \overline{\overline{a_{ij}}} = \frac{1}{144}, \sum b_i^v \overline{a_{ij}} = \frac{1}{720}$$

Seventh order: $\sum b_i^v c_i^6 = \frac{1}{7}, \sum b_i^v c_j^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{840}, \sum b_i^v c_i^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{42}, \sum b_i^v c_i^2 c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{168}, \sum b_i^v c_i\, c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{840}$

$$\sum b_i^v c_j^2 \overline{\overline{a_{ij}}} = \frac{1}{2520}, \sum b_i^v c_i^2 \overline{\overline{a_{ij}}} = \frac{1}{168}, \sum b_i^v c_i\, c_j \overline{\overline{a_{ij}}} = \frac{1}{840}, \sum b_i^v c_j \overline{a_{ij}} = \frac{1}{5040}$$

$$\sum b_i^v c_i \overline{a_{ij}} = \frac{1}{840}, \sum b_i^v a_{ij} = \frac{1}{5040}$$

Eighth order: $\sum b_i^v c_i^7 = \frac{1}{8}, \sum b_i^v c_j^4 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{1680}, \sum b_i^v c_i^4 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{48}, \sum b_i^v c_i^3 c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{192},$

$$\sum b_i^v c_i^2 c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{960}, \sum b_i^v c_i\, c_j^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5760}, \sum b_i^v c_j^3 \overline{\overline{a_{ij}}} = \frac{1}{6720}, \sum b_i^v c_i^3 \overline{\overline{a_{ij}}} = \frac{1}{192},$$

$$\sum b_i^v c_i^2 c_j \overline{\overline{a_{ij}}} = \frac{1}{960}, \sum b_i^v c_i\, c_j^2 \overline{\overline{a_{ij}}} = \frac{1}{5760}, \sum b_i^v c_j^2 \overline{a_{ij}} = \frac{1}{20160},$$

$$\sum b_i^v c_i^2 \overline{a_{ij}} = \frac{1}{960}, \sum b_i^v c_i\, c_j \overline{a_{ij}} = \frac{1}{5760},$$

$$\sum b_i^v c_j\, a_{ij} = \frac{1}{40320}, \sum b_i^v c_i\, a_{ij} = \frac{1}{5760}$$

Ninth Order: $\sum b_i^v c_i^8 = \frac{1}{9}$ [91-133]

2.3 Construction of Three-Stage Eighth Order RKSD Methods

The solution of equations [27-133] using three-stage eighth order RKSD8 methods the finding related to values of $c_i, b_i, b_i', b_i'', b_i''', b_i^{iv}, b_i^v$ for $i = 1,2,3$ are:

$$c_2 = \frac{-3+4c_3}{-4+6c_3}, \quad b_1 = \frac{28c_2c_3-4(c_2+c_3)+1}{20160c_2c_3}, \quad b_2 = \frac{4c_3-1}{20160c_2(c_3-c_2)},$$

$$b_3 = \frac{1-4c_2}{20160c_3(c_3-c_2)}, \quad b_1' = \frac{378c_2c_3-63(c_2+c_3)+18}{45360c_2c_3}, \quad b_2' = \frac{63c_3-18}{45360c_2(c_3-c_2)},$$

$$b_3' = \frac{18-63c_2}{45360c_3(c_3-c_2)}, \quad b_1'' = \frac{15c_2c_3-3(c_2+c_3)+1}{360c_3(c_3-c_2)}, \quad b_2'' = \frac{3c_3-1}{360c_2(c_3-c_2)},$$

$$b_3'' = \frac{-3c_2+1}{360c_3(c_3-c_2)}, \quad b_1''' = \frac{20c_2c_3-5(c_2+c_3)+2}{120c_2c_3}, \quad b_2''' = \frac{5c_3-2}{120c_2(c_3-c_2)},$$

$$b_3''' = \frac{-5c_2+2}{120c_3(c_3-c_2)}, \quad b_1^{iv} = \frac{6c_2c_3-2(c_2+c_3)+1}{12c_2c_3}, \quad b_2^{iv} = \frac{2c_3-1}{12c_2(c_3-c_2)}$$

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$$b_3^{iv} = \frac{1-2c_2}{12c_3(c_3-c_2)}, \quad b_1^{v} = \frac{6c_2c_3-3(c_3+c_2)+2}{6c_2c_3}, \quad b_2^{v} = \frac{3c_3-2}{6c_2(c_3-c_2)},$$

$$b_3^{v} = \frac{2-3c_2}{6c_3(c_3-c_2)}, \quad c_2 = \frac{-3+4c_3}{-4+6c_3},$$

For evaluating minimal value to error norms for eighth order we need to calculate parameters which are free in nature i.e $c_3, b_i, b_i', b_i'', b_i''', b_i^{iv}, b_i^{v}$ for $i = 1,2,3$. Hence the result values of error norms are $\left\|\tau^{(8)}\right\|_2 = 787717 \times 10^{-16}$,$\left\|\tau'^{(8)}\right\|_2 = -417247 \times 10^{-9}$,$\left\|\tau''^{(8)}\right\|_2 = 200905 \times 10^{-9}$, $\left\|\tau'''^{(8)}\right\|_2 = -41094 \times 10^{-9}$, $\left\|\tau^{iv(8)}\right\|_2 = 14671286 \times 10^{-9}$ and $\left\|\tau^{v(8)}\right\|_2 = 1047605 \times 10^{-9}$and $\left\|\tau_g^{(8)}\right\|_2 = 150863684 \times 10^{-9}$.

2.4 Construction of Four-Stage Ninth Order RKSD Methods

The solution of equations [27-133] using four-stage ninth order RKSD9 methods the finding related to values of $c_i, b_i, b_i', b_i'', b_i''', b_i^{iv}, b_i^{v}\ for\ i = 1,2,3,4$ are as:

$$b_1 = \frac{1}{720} - b_2 - b_3 - b_4, \quad b_2 = \frac{\left(\frac{c_3}{4}+\frac{c_4}{4}-c_3c_4-\frac{1}{12}\right)}{5040c_2(c_2-c_3)(c_4-c_2)}, \quad b_3 = \frac{3(c_4+c_2)-1-12c_4c_2}{60480c_3(c_2-c_3)(c_3-c_4)},$$

$$b_4 = \frac{-(1-3(c_2+c_3)+12c_2c_3)}{60480c_4(c_3-c_4)(c_4-c_2)}, \quad b_1' = \frac{1}{120} - b_2' - b_3' - b_4', \quad b_2' = -\frac{[28c_3c_4-8(c_4+c_3)+3}{20160c_2(c_2-c_3)(c_4-c_2)},$$

$$b_3' = \frac{[-28c_2c_4+8(c_4+c_3)-3]}{20160c_3(c_2-c_3)(c_3-c_4)}, \quad b_4' = \frac{-[28c_2c_3-8(c_2+c_3)-3]}{20160c_4(c_3-c_4)(c_4-c_2)}, \quad b_1'' = \frac{1}{24} - b_2'' - b_3'' - b_4'',$$

$$b_2'' = \frac{-[21c_4c_3-7(c_4+c_3)+3]}{2520c_2(c_2-c_3)(c_4-c_2)}, \quad b_3'' = \frac{[-21c_4c_2+7(c_4+c_3)-3]}{2520c_3(c_2-c_3)(c_3-c_4)}, \quad b_4'' = \frac{-[21c_3c_2-7(c_2+c_3)+3]}{2520c_4(c_3-c_4)(c_4-c_2)},$$

$$b_1''' = \frac{1}{6} - b_2''' - b_3''' - b_4''', \quad b_2''' = \frac{-[5c_3c_4-2(c_4+c_3)+1]}{120c_2(c_2-c_3)(c_4-c_2)}, \quad b_3''' = \frac{-[5c_2c_4+2(c_4+c_2)-1]}{120c_3(c_2-c_3)(c_3-c_4)},$$

$$b_4''' = \frac{-[5c_2c_3-2(c_3+c_2)+1]}{120c_4(c_3-c_4)(c_4-c_2)}, \quad b_1^{iv} = \frac{1}{2} - b_2^{iv} - b_3^{iv} - b_4^{iv}, \quad b_2^{iv} = \frac{-[10c_3c_4-5(c_4+c_3)+3]}{60c_2(c_2-c_3)(c_4-c_2)},$$

$$b_3^{iv} = \frac{[-10c_2c_4+5(c_4+c_2)-3]}{60c_3(c_2-c_3)(c_3-c_4)}, \quad b_4^{iv} = \frac{-[10c_2c_3-5(c_3+c_2)+3]}{60c_4(c_3-c_4)(c_4-c_2)}, \quad b_1^{v} = 1 - b_2^{v} - b_3^{v} - b_4^{v},$$

$$b_2^{v} = \frac{-[6c_4c_3-4(c_4+c_3)+3]}{12c_2(c_2-c_3)(c_4-c_2)}, \quad b_3^{v} = \frac{[-6c_2c_4+4(c_4+c_2)-3]}{60c_3(c_2-c_3)(c_3-c_4)}, \quad b_4^{v} = \frac{[6c_2c_3-4(c_3+c_2)+3]}{12c_4(c_3-c_4)(c_4-c_2)},$$

$$c_2 = \frac{(5-24c_3)}{24-90c_3},$$

For evaluating minimal value to error norms for ninth order we need to calculate parameters which are free in nature i.e $c_3, b_i, b_i', b_i'', b_i''', b_i^{iv}, b_i^{v}$ for $i = 1,2,3,4$. Hence the result values of error norms are $\left\|\tau^{(8)}\right\|_2 = 3.3726644 \times 10^{-16}$,$\left\|\tau'^{(8)}\right\|_2 = -1.5319 \times 10^{-9}$,$\left\|\tau''^{(8)}\right\|_2 = -7.65574 \times 10^{-9}$, $\left\|\tau'''^{(8)}\right\|_2 = -2.37144 \times 10^{-9}$,$\left\|\tau^{iv(8)}\right\|_2 = 2.61167 \times 10^{-9}$ and $\left\|\tau^{v(8)}\right\|_2 = 1.35281 \times 10^{-9}$and $\left\|\tau_g^{(8)}\right\|_2 = 3.213598 \times 10^{-9}$.

3. Numerical Result and Conclusion

The efficiency of the discussed RKSD8 and RKSD9 method been well examined with the help of numerical problem.

Problem 1: The homogeneous differential equation given as: $v^{(6)} - 8v^{(3)} = 0$ with initial conditions $v(0) = 0, v'(0) = 1, v''(0) = 1, v'''(0) = 0, v^{iv}(0) = 1, v^{v}(0) = 2$

Solution: The exact solution:

$$v(u) = \frac{1}{24}\left(e^{2u} + 3u(7+3u) - e^{-u}\cos\left(\sqrt{3u}\right)\right)$$

The problem 1 with figure 1 proves the objective of proving the three stage RKSD8 been best as compared to four stage RKSD9 and direct method by computing the maximum global error (Max Error) w.r.t value step function. In other words, the proposed research work will serve as a tool for solving similar form of ODEs $v^{vi} = f(u, v, v'\ v'', v''')$ by taking into account the stability and error analysis of proposed methods as compared to traditional methods. Also from numerical results, the best outcome been received about less number of function evaluations for both RKSD8 and RKSD9 methods as compared to other existing RK methods.

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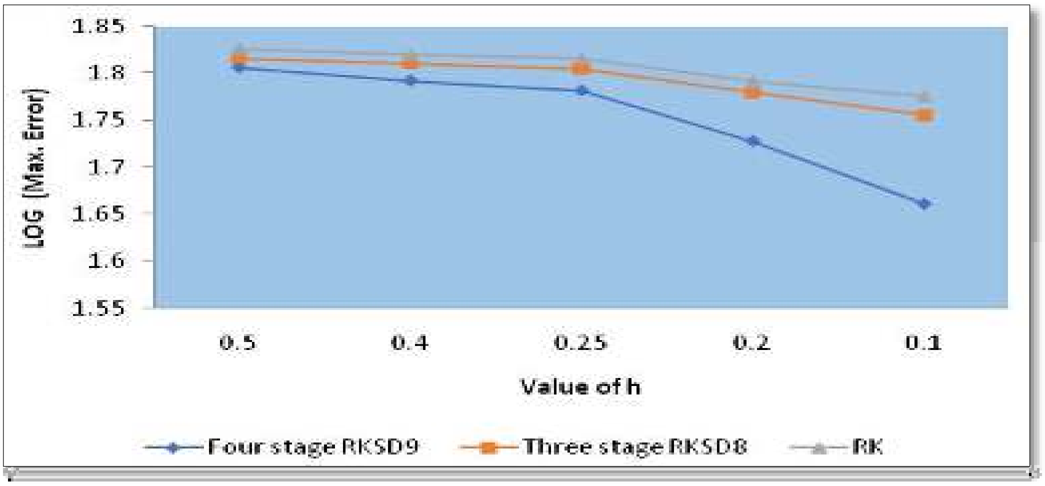

Figure 1: The efficiency curves of three stage RKSD8, four stage RKSD9 and RK method

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